MATH 53H - Solutions to Problem Set III

1. Fix λ not an eigenvalue of L. Then $\det(\lambda I - L) \neq 0 \Rightarrow \lambda I - L$ is invertible. We have then

$$p_{L+N}(\lambda) = \det(\lambda I - L - N) = \det(\lambda I - L)\det(I - (\lambda I - L)^{-1}N) =$$

$$= p_L(\lambda) \cdot \det(I - (\lambda I - L)^{-1}N)$$

Since L and N commute and N is nilpotent, $(\lambda I - L)^{-1}$ commutes with N and thus their product is a nilpotent matrix, which we denote by B. Since B is nilpotent, its only eigenvalue is 0 (why?) and hence

$$p_B(\lambda) = \lambda^n \Rightarrow \det(I - (\lambda I - L)^{-1}N) = \det(I - B) = p_B(1) = 1$$

We obtain from the above $p_{L+N}(\lambda) = p_L(\lambda)$ for all λ which are not an eigenvalue of L. Since there are infinitely many such λ and p_{L+N}, p_L are polynomials, they have to be equal, as we want.

Remark From this result, we can show that the dimensions of the generalized eigenspaces equal the multiplicities of the eigenvalues, since L is defined by multiplication by the corresponding eigenvalue on each generalized eigenspace. We use this fact in the next Exercise.

2. We show first that that if λ is an eigenvalue of A with multiplicity ν, then e^λ is one for $\exp(A)$ with multiplicity at most ν:
If $Ax = \lambda x$ then

$$\exp(A)x = \sum_{n=0}^{\infty} \frac{1}{n!} A^n x = \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^n x = e^\lambda x$$

thus e^λ is also an eigenvalue. Suppose now that x is a generalized eigenvector of A for λ, i.e. $(A - \lambda I)^\nu x = 0$. From the definition of $\exp(A)$ it is easy to see that $\exp(A) - e^\lambda I = B(A - \lambda I)$ for a matrix B that commutes with A. Therefore

$$(\exp(A) - e^\lambda I)^\nu x = B^\nu (A - \lambda I)^\nu x = 0$$

implying that x is a generalized eigenvector of $\exp(A)$ for e^λ. We have shown that $\ker(A - \lambda I)^\nu \subseteq \ker(\exp(A) - e^\lambda I)^\nu$ and hence

$$\mathbb{C}^n = \bigoplus_{i=0}^{m} \ker(A - \lambda_i I)^{\nu_i} \subseteq \bigoplus_{i=0}^{m} \ker(\exp(A) - e^{\lambda_i} I)^{\nu_i} \subseteq \mathbb{C}^n \Rightarrow$$

$$\mathbb{C}^n = \bigoplus_{i=0}^{m} \ker(\exp(A) - e^{\lambda_i} I)^{\nu_i} \ (*)$$

Note that the right hand side is indeed a direct sum decomposition, since the e_{λ_i} are distinct.
From the above equality it also follows that

$$\dim\ker(A - \lambda_i I)^{\nu_i} = \dim\ker(\exp(A) - e^{\lambda_i} I)^{\nu_i} = \nu_i$$

for all i. This observation shows that $(*)$ is the decomposition into generalized eigenspaces of $\exp(A)$ and therefore

$$p_{exp(A)}(\lambda) = (\lambda - e^{\lambda_1})^{\nu_1} \cdots (\lambda - e^{\lambda_m})^{\nu_m}$$

3. Consider

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

We have

$$\exp(tA) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{pmatrix}$$

and hence the general solution is

$$x(t) = \exp(tA)x(0) = \begin{pmatrix} c_1 \\ c_2 \cos t + c_3 \sin t \\ -c_2 \sin t + c_3 \cos t \end{pmatrix}$$

with $x(0) = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$.
It is clear that all non-constant solutions ($c_2 \neq 0$ or $c_3 \neq 0$) are periodic with period 2π.

4. In Exercise 5 in Problem Set 2, we have computed

$$\exp(tA) = \begin{pmatrix} \cos t & \sin t & t\cos t & t\sin t \\ -\sin t & \cos t & -t\sin t & t\cos t \\ 0 & 0 & \cos t & \sin t \\ 0 & 0 & -\sin t & \cos t \end{pmatrix}$$

Hence the general solution with initial condition $x(0) = c = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}$ is

$$x(t) = \exp(tA)x(0) = \begin{pmatrix} c_1 \cos t + c_2 \sin t + t(c_3 \cos t + c_4 \sin t) \\ -c_1 \sin t + c_2 \cos t + t(-c_3 \sin t + c_4 \cos t) \\ c_3 \cos t + c_4 \sin t \\ -c_3 \sin t + c_4 \cos t \end{pmatrix}$$

From this expression it is easy to see that $x(t)$ is bounded if and only if $c_3 = c_4 = 0$, i.e. c belongs to a two-dimensional plane inside $\mathbb{R}^4$.

5. (a) We have $p_A(\lambda) = ((\lambda - a)^2 + b^2)(\lambda - b)$ and thus the eigenvalues are $a \pm bi, b$ with corresponding eigenvectors $v_1 = \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 0 \\ -i \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.
Therefore the general solution is a linear combination of the vectors

$$\Re(e^{(a+ib)t}v_1) = \begin{pmatrix} e^{at} \cos bt \\ 0 \\ -e^{at} \sin bt \end{pmatrix}, \Im(e^{(a+ib)t}v_1) = \begin{pmatrix} e^{at} \sin bt \\ 0 \\ e^{at} \cos bt \end{pmatrix}, e^{bt}v_3 = \begin{pmatrix} 0 \\ e^{bt} \\ 0 \end{pmatrix}$$

Hence it is given as

$$x(t) = \begin{pmatrix} e^{at}(c_1 \cos bt + c_2 \sin bt) \\ c_3 e^{bt} \\ e^{at}(-c_1 \sin bt + c_2 \cos bt) \end{pmatrix}$$

Note that this answer is valid even when $b = 0$ or $a = 0$, in which case A has repeated eigenvalues.

(b) All solutions converge to constant or periodic solutions as $t \to +\infty$ when both exponentials are constant or decaying, i.e. when $a \leq 0$ and $b \leq 0$.

(c) For all non-zero solutions to diverge, both exponentials must diverge at $+\infty$, i.e. $a > 0$ and $b > 0$.

(d) This is the same as (b). If $a > 0$ or $b > 0$ then almost all solutions diverge, e.g. for $a > 0$ all solutions with $c_1 \neq 0$ or $c_2 \neq 0$ diverge and this is an open and dense subset of the space $\mathbb{R}^3$ of the initial conditions.

(e) By the above, it is easy to see that almost all (and not all) solutions diverge when $a > 0, b \leq 0$ or $a \leq 0, b > 0$.

We can observe that (b) and (d) yield the same answer (in some sense convergence is a closed condition here) and that the four quadrants exhibit different behaviours moving from convergence to divergence of almost all solutions and then divergence of all non-zero solutions.

6. **(a)** We introduce auxiliary variables $y_1 = x_1', y_2 = x_2'$ and let

$$X = \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix}$$

We can then rewrite the system equivalently in the form $X' = AX$ where

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -(k_1 + k_2) & k_2 & 0 & 0 \\ k_2 & -(k_1 + k_2) & 0 & 0 \end{pmatrix}$$

(b) We can compute $p_A(\lambda) = (\lambda^2 + {\omega_1}^2)(\lambda^2 + {\omega_2}^2)$ where $\omega_1 = \sqrt{k_1}, \omega_2 = \sqrt{k_1 + 2k_2}$. Hence the eigenvalues are $\pm i\omega_1, \pm i\omega_2$ with corresponding eigenvectors

$$\begin{pmatrix} 1 \\ 1 \\ \pm i\omega_1 \\ \pm i\omega_1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ \pm i\omega_2 \\ \mp i\omega_2 \end{pmatrix}$$

(c) Looking at the first two coordinates of X we have that $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ is a linear combination of

$$\Re(e^{i\omega_1 t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}), \Im(e^{i\omega_1 t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}), \Re(e^{i\omega_2 t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}), \Im(e^{i\omega_2 t} \begin{pmatrix} 1 \\ -1 \end{pmatrix})$$

and therefore the general solution is given by

$$x_1(t) = A\cos(\omega_1 t) + B\sin(\omega_1 t) + C\cos(\omega_2 t) + D\sin(\omega_2 t)$$

$$x_2(t) = A\cos(\omega_1 t) + B\sin(\omega_1 t) - C\cos(\omega_2 t) - D\sin(\omega_2 t)$$

(d) By re-arranging the formula of the general solution we may write equivalently

$$x_1(t) = A_1\cos(\omega_1 t - \phi_1) + A_2\cos(\omega_2 t - \phi_2)$$

$$x_2(t) = A_1\cos(\omega_1 t - \phi_1) - A_2\cos(\omega_2 t - \phi_2)$$

If $A_2 = 0$ then we see that $x(t)$ is periodic with period $T = \frac{2\pi}{\omega_1}$. Similarly, if $A_1 = 0$, then $x(t)$ is periodic with period $T = \frac{2\pi}{\omega_2}$ and clearly if both are zero, then $x(t)$ is constant, being identically zero.
If $A_1, A_2 \neq 0$ then $x(t)$ is periodic if and only if $\frac{\omega_2}{\omega_1} \in \mathbb{Q}$. If $x(t)$ is periodic with period T then $\cos(\omega_1 t - \phi_1) = \frac{1}{2A_1}(x_1(t) + x_2(t))$ and $\cos(\omega_2 t - \phi_2) = \frac{1}{2A_2}(x_1(t) - x_2(t))$ are also T-periodic. This implies that $T = n\frac{2\pi}{\omega_1} = m\frac{2\pi}{\omega_2}$ for integers m and n and hence $\frac{\omega_2}{\omega_1} = \frac{m}{n} \in \mathbb{Q}$.
Conversely, if $\frac{\omega_2}{\omega_1} = \frac{m}{n} \in \mathbb{Q}$ then it is clear that $x(t)$ is periodic with period $T = n\frac{2\pi}{\omega_1} = m\frac{2\pi}{\omega_2}$.

7. From what we have seen, we know that

$$\exp(tA) = \begin{pmatrix} \cos\omega_1 t & \sin\omega_1 t & 0 & 0 & 0 & 0 \\ -\sin\omega_1 t & \cos\omega_1 t & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos\omega_2 t & \sin\omega_2 t & 0 & 0 \\ 0 & 0 & -\sin\omega_2 t & \cos\omega_2 t & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-t} & 0 \\ 0 & 0 & 0 & 0 & 0 & e^t \end{pmatrix}$$

and the general solution is given by

$$x(t) = \begin{pmatrix} c_1\cos\omega_1 t + c_2\sin\omega_1 t \\ -c_1\sin\omega_1 t + c_2\cos\omega_1 t \\ c_3\cos\omega_2 t + c_4\sin\omega_2 t \\ -c_3\sin\omega_2 t + c_4\cos\omega_2 t \\ c_5 e^{-t} \\ c_6 e^t \end{pmatrix}$$

where

$$x(0) = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{pmatrix}$$

We will make use of the discussion of Exercise 6.(d) above. We have:

(a) Since $x_6(t) = 0$ and $x_5(t) \to 0$ as $t \to +\infty$, $x(t)$ converges to a bounded, non-periodic solution, which is actually dense inside a torus in $\mathbb{R}^4$. For a nice, more detailed discussion of this, you can look at Section 6.2 (Harmonic Oscillators) of Hirsch, Smale and Devaney.

(b) Here $x_6(t)$ diverges, while the first four coordinates exhibit the same behaviour as in (a), so $x(t)$ is divergent.

(c) Here $x_6(t)$ diverges, while the first two coordinates give a periodic solution (a circle) of period $\frac{2\pi}{\omega_1}$, so $x(t)$ is divergent.

(d) In this case, the solution is periodic (a circle inside a two-plane inside $\mathbb{R}^6$) with period $\frac{2\pi}{\omega_1}$.